Approximation Operators, Exponential, q-Exponential, and Free Exponential Families

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Abstract

Using the technique developed in approximation theory, we construct examples of exponential families of infinitely divisible laws which can be viewed as ε-deformations of the normal, gamma, and Poisson exponential families. Replacing the differential equation of approximation theory by a q-differential equation, we define the q-exponential families, and we identify all q-exponential families with quadratic variance functions when $|q| < 1$. We elaborate on the case of $q = 0$ which is related to free convolution of measures. We conclude by considering briefly the case $q > 1$, and other related generalizations.

Running Title. Exponential Families

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1 Introduction

1.1 Exponential Type Approximation Operators

C. P. May [30] introduced exponential type operators as

$$S_\lambda(f)(m) = \int_{\mathbb{R}} W_\lambda(m,u)\, f(u)du, \tag{1.1}$$

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where W_λ is a generalized function satisfying the generalized differential equation

$$\frac{\partial W_\lambda}{\partial m} = \lambda W_\lambda \frac{u-m}{v(m)}, \quad \lambda > 0, \tag{1.2}$$

and v is a polynomial of degree at most 2. Moreover May assumed that S_λ is a positive operator and that

$$\int_{\mathbb{R}} W_\lambda(m,u)\, du = 1. \tag{1.3}$$

May knew the exact form of S_λ in all possible cases except when v has two non-real zeros. May proved that S_λ and certain linear combinations of it approximate continuous functions in the sense that $\lim_{\lambda\to\infty} S_\lambda(f,m) = f(m)$. Later Ismail and May [23] extended the approximation theoretic study to the case when v is an analytic and strictly positive function on (A,B), a component of $\{t: \ v(t) > 0\}$. They also identified W_λ when $v(m) = 1+m^2$, the general case with two complex roots.

In the above mentioned work, it was observed that

$$\int_{\mathbb{R}} W_\lambda(m,u)\, u\, du = m, \qquad \int_{\mathbb{R}} W_\lambda(m,u)\, (u-m)^2 du = \frac{v(m)}{\lambda} \tag{1.4}$$

follow from (1.2) and (1.3). Hence m and $v(m)/\lambda$ are the mean and variance of $W_\lambda(m,u)$, respectively.

The parameter λ is important in approximation theory since as $\lambda\to\infty$ the variance tends to zero and W_λ becomes a unit atomic measure concentrated at $u=m$. Ismail and May [23] observed that the differential equation (1.2) has at most one solution which satisfies the normalization (1.3) and makes S_λ a positive operator. They used the notation

$$q(m) = \int_c^m \frac{d\theta}{v(\theta)}, \quad c\in(A,B), \qquad g(q(m)) = q(g(m)) \equiv m. \tag{1.5}$$

Moreover Ismail and May proved that

$$S_\lambda(f,m) = \int_{\mathbb{R}} C_\lambda(u) \exp\left(-\lambda \int_c^m \frac{\theta-u}{v(\theta)}\, d\theta\right) f(u)\, du \tag{1.6}$$

and the function (or generalized function) $C_\lambda(u)$ is computed by inverting the Laplace transform

$$\exp\left(\lambda \int_c^m \frac{\theta}{v(\theta)}\, d\theta\right) = \int_{\mathbb{R}} C_\lambda(u) \exp\left(\lambda u \int_c^m \frac{d\theta}{v(\theta)}\right) du.$$

The above formula is

$$\exp\left(\lambda \int_c^{g(z)} \frac{\theta}{v(\theta)}\, d\theta\right) = \int_{\mathbb{R}} C_\lambda(u) \exp(\lambda u z) du, \tag{1.7}$$

and is valid for Re $z \in$ Range of $q(m), m \in (A, B)$. (Compare [29, (2.1)].) The theory of bilateral Laplace transform is in [40].

Ismail [20] considered the case when $v(m)$ has a simple zero at an end point, which without loss of generality is taken as $m = 0$. He used the notation

$$h(z) := \frac{1}{v(z)} - \frac{1}{z},$$

(1.8)
$$\xi = \xi(m) := \frac{m}{c} \exp\left\{ \int_c^m h(\theta) d\theta \right\}, \quad \eta(\xi) := m - c + \int_c^m \theta h(\theta) d\theta.$$

He further assumed that $h(z)$ is analytic at $z = 0$ and $\eta'(0) \neq 0$. In his notation W_λ is a discrete probability distribution and takes the form

$$W_\lambda(m, du) = \sum_{n=0}^{\infty} \phi_n(\lambda) \exp\left(- \int_c^m \frac{\lambda\theta - n}{v(\theta)} \, d\theta \right) \delta_{n/\lambda}(du), \tag{1.9}$$

where $\{\phi_n : n = 0, 1, \dots\}$ are generated by

$$\exp(\lambda \eta(\xi)) = \sum_{n=0}^{\infty} \phi_n(\lambda) \, \xi^n, \tag{1.10}$$

and $\delta_a(du)$ is a unit atomic measure concentrated at $a \in \mathbb{R}$. Ismail also showed that W_λ in (1.9) is independent of the choice of $c \in (A, B)$.

1.2 Exponential Families

Fix a positive non-degenerate σ-finite measure μ on $\mathbb{R}$ with the property that

$$L(\theta) = \int_{\mathbb{R}} \exp(\theta u) \mu(du) < \infty$$

for all $C < \theta < D$. Denote

$$\kappa(\theta) = \ln L(\theta).$$

The exponential family generated by μ is the set of probability measures

$$\mathcal{F}(\mu) := \{ P_\theta(du) = \exp(\theta u - \kappa(\theta)) \mu(du) : \theta \in (C, D) \}.$$

For a concise introduction, see [24, Chapter 2]. Most authors take θ from the largest admissible interval; ref. [29] restricts θ to a maximal open interval.

This family can be conveniently re-parameterized by the mean. Since μ is non-degenerate, $\kappa(\cdot)$ is strictly convex so that $\kappa'()$ is strictly increasing on (C, D); it is also clear that κ is analytic on (C, D). Let

$$A = \lim_{\theta \to C^+} \kappa'(\theta), \; B = \lim_{\theta \to D^-} \kappa'(\theta). \tag{1.11}$$

Clearly, $\kappa' : (C, D) \to (A, B)$ is invertible, and $m = \kappa'(\theta) = \int_{\mathbb{R}} u P_\theta(du) \in (A, B)$. So for $\theta \in (C, D)$ probability measure P_θ is determined uniquely by its

mean $m \in (A, B)$. Let ψ be the inverse function to κ', i.e. $\kappa'(\psi(m)) = m$ and $\psi(\kappa'(\theta)) = \theta$ for all $m \in (A, B)$, $\theta \in (C, D)$. Then the probability measures

$$W(m, du) := P_{\psi(m)}(du),\ m \in (A, B) \tag{1.12}$$

provide another parametrization of $\mathcal{F}(\mu)$. Since

$$\int_{\mathbb{R}} uW(m, du) = m,$$

this is parametrization by the means. The variance function $V : (A, B) \to \mathbb{R}$ is now defined as

$$V(m) = \int (u - m)^2 W(m, du),$$

compare (1.4). Notice that $V(m) = \kappa''(\psi(m))$. It is known that the variance function V together with (A, B) determines μ uniquely, see [24, Theorem 2.11], [31, page 67], or [29, Proposition 2.2].

1.3 Exponential Families and Exponential Operators

The connection between exponential families and exponential operators has been noticed in [12, Section 5], see also [38, Theorem 2]. Here we give a somewhat more precise version of this relation that allows for parameter $\lambda > 0$ thus connecting exponential operators with dispersion models [24].

Suppose that a non-degenerate σ-finite measure μ with exponential moments of order $\theta \in (C, D)$ generates exponential family with the variance function $V(m)$, $m \in (A, B)$. For natural $\lambda = 1, 2, \dots$ denote by μ_λ the λ-dilation of the convolution power $\mu^{*\lambda}$, i.e. $\mu_\lambda(U) := (\mu * \mu * \cdots * \mu)(\lambda U)$. The natural exponential family generated by μ_λ is the family of measures

$$\mathcal{F}(\mu_\lambda) := \{P_{\lambda,\theta}(du) = \exp(-\theta u - \kappa_\lambda(\theta))\mu_\lambda(du) :\ \theta \in (C\lambda, D\lambda)\},$$

where $\kappa_\lambda(\theta) = \lambda\kappa(\theta/\lambda)$. In particular, $\psi_\lambda(m)$ which is the inverse of $\kappa'_\lambda(\theta)$ is $\psi_\lambda(m) = \lambda\psi(m)$ and the new variance function is

$$V_\lambda(m) = \kappa''_\lambda(\psi_\lambda(m)) = \frac{V(m)}{\lambda}. \tag{1.13}$$

Notice that since $\kappa'_\lambda(\theta) = \kappa'(\theta/\lambda)$, the limits in (1.11) do not depend on λ. Parameterized by the mean, the family is

$$\mathcal{F}(\mu_\lambda) = \{W_\lambda(m, du) :\ m \in (A, B)\}.$$

We now verify that these measures satisfy equation (1.2).

Proposition 1.1. *If a positive non-degenerate σ-finite measure μ with exponential moments of order $\theta \in (C, D)$ generates the natural exponential family with the variance function $V(m)$ defined for $m \in (A, B)$, then for natural λ*

measures μ_λ *generate the natural exponential family* $W_\lambda(m, du)$ *such that the corresponding integral operators*

$$S_\lambda(f)(m) = \int f(u) W_\lambda(m, du)$$

are the exponential type operators which satisfy equation (1.2) with $\lambda = 1, 2, \ldots$ *and* $v(m) = V(m)$ *for* $m \in (A, B)$.

Proof. It is straightforward to verify that (1.2) holds with $v(m) = V(m)$, $\lambda = 1, 2, \ldots$. Since

$$S_\lambda(f)(m) = \int f(u) \exp(\psi_\lambda(m) u - \kappa_\lambda(\psi_\lambda(m))) \mu_\lambda(du),$$

differentiating under the integral sign we get

$$\int f(u) \frac{\partial}{\partial m} W_{\lambda,m}(du) \\ = \int f(u) \psi'_\lambda(m)(u - \kappa'_\lambda(\psi_\lambda(m))) \exp\left(\psi(m) u - \kappa(\psi(m))\right) \mu_\lambda(du).$$

As $\kappa'_\lambda(\psi_\lambda(m)) = m$ and $\psi'_\lambda(m) = 1/\kappa''_\lambda(\psi_\lambda(m)) = 1/V_\lambda(m) = \lambda/V(m)$, (1.2) follows. □

Remark 1.2. If equation (1.2) has solution $S_\lambda(f, m)$ for all $0 < \lambda \leq 1$, and $m \in (A, B)$ then the exponential family generated by μ consists of infinitely divisible probability laws.

Proof. To prove infinite divisibility, without loss of generality we may concentrate on fixed $W_1(m_0, \mu) \in \mathcal{F}(\mu)$. It is well know that with the range of means (A, B) kept fixed, $\mathcal{F}(\mu) = \mathcal{F}(W_1(m_0, \mu))$, see [24, Exercise 2.12].

For $\lambda = 1/k$ where $k = 1, 2, \ldots$, let $W_\lambda(m, du), m \in (A, B)$ be the solution of (1.2). The variance function is $V(m)/\lambda = kV(m)$. Denote by ν the dilation of measure $W_\lambda(m_0, du)$ by k. By (1.13), the exponential family $\mathcal{F}(\nu^{*k})$ has the same variance function $V(m)$ as the exponential family $\mathcal{F}(W_1(m_0, du))$. By uniqueness of parametrization by the means, $W_1(m_0, du) = \nu^{*k}(du)$, so infinite divisibility follows. □

Proposition 1.1 shows that the celebrated result [31, Section 4] can be derived as a consequence of [23, Theorem 3.3]; the latter paper contains also several cubic variance functions and other interesting examples. Another interesting result [29, Proposition 4.4] is a consequence of [20, Theorem 3.8].

1.4 Notation

We shall follow the terminology in [13] for hypergeometric functions, namely that

$$(1.14)\qquad \begin{aligned} &(a)_n := 1, \qquad (a)_n = \prod_{j=0}^{n-1}(a+j),\\ &{}_2F_1\left(\begin{array}{c} a,b \\ c \end{array} \middle| \, z \right) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\;n!}\, z^n. \end{aligned}$$

The modified Bessel functions are [14]

$$(1.15)\quad I_\nu(z) := \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{n!\,\Gamma(n+\nu+1)}, \quad K_\nu(z) := \frac{\pi}{2}\,\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)}.$$

The Lagrange expansion theorem [33, (L), page 145] says that if $f(z)$, $\phi(z)$ are analytic in a neighborhood of $z = 0$, $\phi(0) \neq 0$ and $\xi := m/\varphi(m)$ then

$$(1.16)\qquad f(m(\xi)) = f(0) + \sum_{n=1}^{\infty} \frac{\xi^n}{n!} \left[\frac{d^{n-1} f'(x)[\phi(x)]^n}{dx^{n-1}} \right]_{x=0}.$$

By $1_{(a,b)}(u)$ we denote the indicator function of (a, b).

Occasionally, we also use the q-notation

$$\begin{aligned} (a;q)_n &:= \prod_{k=0}^{n-1}(1 - aq^k),\\ (a;q)_\infty &:= \prod_{k=0}^{\infty}(1 - aq^k),\\ (a_1, a_2, \ldots, a_m; q)_\infty &:= (a_1;q)_\infty (a_2;q)_\infty \ldots (a_m;q)_\infty,\\ [n]_q &:= 1 + q + \cdots + q^{n-1},\\ [n]_q! &:= [1]_q[2]_q \ldots [n]_q = \frac{(q;q)_n}{(1-q)^n},\\ \left[\begin{array}{c} n \\ k \end{array} \right]_q &:= \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{(q;q)_k (q;q)_{n-k}}{(q;q)_n}, \end{aligned}$$

with the usual conventions $[0]_q = 0, [0]_q! = 1$. Most of this notation is taken from [17].

2 Examples of Variance Functions

2.1 ε-Deformations of Quadratic Variance Functions

Letac and Mora [29, page 3] raise the question of classifying exponential families with variances functions of the form

$$(2.1)\qquad P(m) + Q(m)\sqrt{R(m)},$$

where P, Q, R are polynomials of degree at most $3, 2, 2$ respectively. Letac [28, page 74] initiated the study of variance functions (2.1) when P is a multiple of R. The latter class was investigated by Kokonendji [27] who also gave an excellent overview of other known cases. Kokonendji [26] used probabilistic techniques to investigate variance functions in the Seshadri's class $V(m) = \sqrt{R(m)}P(\sqrt{R(m)})$. This section further advances the investigation of the variance functions (2.1).

We use Proposition 1.1 to identify certain exponential families $\mathcal{F}_\varepsilon$ with the variance function of the form

$$V(m) = (am^2 + bm + c)\sqrt{1 + \varepsilon m^2},\ \varepsilon > 0. \tag{2.2}$$

These are ϵ-deformations of the quadratic variance family $\mathcal{F}_0$ analyzed in [23] and [31]. From Mora's theorem [24, Theorem 2.12], as $\varepsilon \to 0$ while (A, B) is fixed, the corresponding probability laws in $\mathcal{F}_\varepsilon$ weakly converge to the respective laws in $\mathcal{F}_0$.

We also give two examples of the functions (2.1) which are not the variance functions.

2.1.1 Continuous Exponential Families

In this section we consider the following continuous ε-deformations:

(i) ε-Gaussian family $V(m) = (1 + \varepsilon m^2)\sqrt{1 + \varepsilon m^2}$,

(ii) ε-gamma family $V(m) = m^2\sqrt{1 + \varepsilon m^2}$.

The first case (i) gives an infinitely divisible family introduced in [8], see [27, Example 2.5] and [24, Exercise 3.2].

Theorem 2.1 (Kokonendji [27]). *For $\lambda > 0$, $\varepsilon > 0$, the exponential family with the variance function*

$$V(m) = \frac{1}{\lambda}(1 + \varepsilon m^2)\sqrt{1 + \varepsilon m^2},\ m \in \mathbb{R}$$

consists of the infinitely divisible probability laws with the densities

$$\exp\left(\frac{\lambda}{\varepsilon}\left(\frac{1 + um\varepsilon}{\sqrt{1 + \varepsilon m^2}} - 1\right)\right) \frac{\lambda}{\pi\varepsilon\sqrt{1 + \varepsilon u^2}}\ K_1\left(\frac{\lambda}{\varepsilon}\sqrt{1 + \varepsilon u^2}\right). \tag{2.3}$$

Before we give a proof of Theorem 2.1 we show how we give a formal argument. In the present case we have

$$c = 0,\quad q(m) = \frac{m}{\sqrt{1 + \varepsilon m^2}},\quad g(z) = \frac{z}{\sqrt{1 - \varepsilon z^2}},\quad \int_0^m \frac{\theta\, d\theta}{v(\theta)} = 1/\varepsilon - \frac{1}{\varepsilon\sqrt{1 + \varepsilon m^2}}.$$

Now (1.7), after $z \mapsto \sqrt{\varepsilon}z/\lambda$, $u \mapsto u/\sqrt{\varepsilon}$, and $\lambda \mapsto \lambda\varepsilon$ becomes

$$\exp\left(\lambda - \sqrt{\lambda^2 - z^2}\right) = \int_{\mathbb{R}} \exp(uz)\, C_\lambda(u) du/\sqrt{\varepsilon}, \quad \operatorname{Re} z \in (-\lambda, \lambda). \tag{2.4}$$

If we know that the left-hand side of the above equation is a bilateral Laplace transform we can use the inversion theorem, Theorem 5a on page 241 of Widder [39, §6.5], and see that

$$\begin{aligned} e^{-\lambda} C_\lambda(-u)/\sqrt{\varepsilon} &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp(-\sqrt{\lambda^2 - v^2})\, \exp(uv)\, dv \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \exp(-\sqrt{\lambda^2 + v^2})\, \cos(uv)\, dv. \end{aligned}$$

Formula (26), page 16 of [15] implies

$$C_\lambda(u) = \frac{\lambda\, e^{\lambda}\sqrt{\varepsilon}}{\pi\,\sqrt{1+u^2}} K_1(\lambda\sqrt{1+u^2}). \tag{2.5}$$

Proof of Theorem 2.2. We verify (2.4) directly. With the above $C_\lambda(u)$ the right-hand side of (2.4) is

$$\frac{\lambda\, e^{\lambda}}{\pi}\int_0^{\infty} \cosh(uz)\, \frac{K_1(\lambda\sqrt{1+u^2})}{\sqrt{1+u^2}}\, du. \tag{2.6}$$

In view of [14, (7.2.40)]

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x, \quad K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x},$$

we apply [14, (7.14.46)] and conclude that the expression in (2.6) equals the left-hand side of (2.4). Substituting back the original values of λ, z, u we get (2.3). □

We now consider case (ii), which yields the infinitely divisible distributions from [16, page 58].

Theorem 2.2 (Letac [28, page 46, Example 8.2]). *For $\lambda > 0$, $\varepsilon > 0$, the natural exponential family with the variance function*

$$V(m) = \frac{m^2}{\lambda}\sqrt{1+\varepsilon m^2}$$

defined on $m > 0$, consists of the absolutely continuous infinitely divisible probability laws

$$\left(\frac{1+\sqrt{1+\varepsilon m^2}}{\sqrt{\varepsilon} m}\right)^{\lambda} \frac{\lambda}{u}\, I_\lambda(\sqrt{\varepsilon} u) \exp\left(-\frac{\lambda u(\sqrt{1+\varepsilon m^2})}{m}\right) 1_{(0,\infty)}(u)\, du. \tag{2.7}$$

Proof. We choose $c = 1/\sqrt{\varepsilon}$ and apply

$$\int_c^m \frac{\theta}{v(\theta)} d\theta = \ln\left(\frac{m\,\sqrt{\varepsilon}(1+\sqrt{2}\,)}{1+\sqrt{1+\varepsilon m^2}}\right),$$

$$\int_c^m \frac{d\theta}{v(\theta)} = \sqrt{2\varepsilon} - \frac{\sqrt{1+\varepsilon m^2}}{m}.$$

Therefore (1.7) gives

$$\left(\frac{m\sqrt{\varepsilon}(1+\sqrt{2}\,)}{1+\sqrt{1+\varepsilon m^2}}\right)^{\lambda} = \int_0^\infty C_\lambda(u)\, e^{\lambda\sqrt{2\varepsilon}\,u} \exp\left(-\lambda u(\sqrt{1+\varepsilon m^2})/m\right)\, du.$$

To invert the above Laplace we set $w = (\sqrt{1+\varepsilon m^2})/m$ so that $m = 1/\sqrt{w^2-\varepsilon}$. Thus for $w > \sqrt{\varepsilon}$ we need to invert

$$\int_0^\infty C_\lambda(u)\, e^{\lambda\sqrt{2\varepsilon}\,u} \exp(-\lambda u w)\, du = \varepsilon^{\lambda/2}(1+\sqrt{2})^{\lambda} \left(w+\sqrt{w^2-\varepsilon}\right)^{-\lambda}.$$

We use (28), page 240 in [15] to invert the above Laplace transform and establish (2.7). □

2.1.2 Discrete Exponential Families

In this section we consider the following cases:

(i) the ε-deformation of the Poisson family $V(m) = m\sqrt{1+\varepsilon m^2}$,

(ii) the discrete ε-deformation of the Gaussian family $V(m) = \sqrt{1+\varepsilon m^2}$.

We first consider case (i). In this case $B = +\infty$ and we choose $c = 1/\sqrt{\varepsilon}$. It is a calculus exercise to derive

$$\int_1^m h(\theta)\, d\theta = \ln\left(\frac{1+\sqrt{2}}{1+\sqrt{1+m^2}}\right),$$

$$\int_1^m \theta h(\theta)\, d\theta = \ln\left(\frac{m+\sqrt{1+m^2}}{1+\sqrt{2}}\right) + 1 - m.$$

Hence

$$\xi(m) = \frac{(1+\sqrt{2})\sqrt{\varepsilon}m}{1+\sqrt{1+\varepsilon m^2}}, \qquad \eta(\xi(m)) = \frac{1}{\sqrt{\varepsilon}}\ln\left(\frac{\sqrt{\varepsilon}m+\sqrt{1+\varepsilon m^2}}{1+\sqrt{2}}\right). \tag{2.8}$$

With $\zeta(m) = \xi(m)/(1+\sqrt{2})$ it follows that $m = \frac{2\zeta}{\sqrt{\varepsilon}(1-\zeta^2)}$, so that

$$\lambda\eta(\xi(m)) = \ln\left(\frac{1+\zeta}{1-\zeta}\right)^{\lambda/\sqrt{\varepsilon}} - \ln(1+\sqrt{2})^{\lambda/\sqrt{\varepsilon}}. \tag{2.9}$$

A simple calculation shows that

$$\left(\frac{1+\zeta}{1-\zeta}\right)^{\lambda} = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, & \lambda \\ -\lambda-n+1 & \end{matrix}\,\middle|\, -1\right) \zeta^n.$$

This proves the following theorem.

Theorem 2.3 (Letac [28, pg 98, (3)]). *For $\lambda > 0$, $\varepsilon > 0$, the exponential family with the variance function*

$$V(m) = \frac{m}{\lambda}\sqrt{1+\varepsilon m^2}$$

defined on $m > 0$, consists of infinitely divisible discrete probability measures

$$\begin{aligned}(2.10)\quad &\left(\sqrt{\varepsilon}m + \sqrt{1+\varepsilon m^2}\right)^{-\lambda/\sqrt{\varepsilon}} \sum_{n=0}^{\infty} \frac{(\lambda/\sqrt{\varepsilon})_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, & \lambda/\sqrt{\varepsilon} \\ -\lambda/\sqrt{\varepsilon} - n + 1 & \end{matrix} \middle| -1 \right) \\ &\times \left(\frac{\sqrt{\varepsilon}m}{1+\sqrt{1+\varepsilon m^2}} \right)^n \delta_{n/\lambda}(du).\end{aligned}$$

We now consider **Case (ii)**. This is again a known case: [24, Exercise 3.15] gives an answer in terms of the compound Poisson law, [27, Example 2.6] writes the answer in terms of $\sum_{k\in Z} I_k(\lambda/\varepsilon)\delta_k(du)$. We remark that this is an example of a discrete indefinitely divisible natural family to which [29, Proposition 4.4] or [20, Theorem 3.3] cannot be applied.

Theorem 2.4 (Letac [28, page 100, (8)]). *For $\lambda > 0$, $\varepsilon > 0$, the natural exponential family with the variance function*

$$V(m) = \frac{1}{\lambda}\sqrt{1+\varepsilon m^2}$$

defined on $m > 0$, consists of infinitely divisible discrete probability measures

(2.11)
$$e^{-\frac{\lambda}{\varepsilon}\sqrt{1+\varepsilon m^2}} \sum_{n=0}^{\infty} \frac{\lambda^n}{(2\varepsilon)^n n!} \sum_{k=0}^{n} \binom{n}{k} \left(\sqrt{\varepsilon}m + \sqrt{1+\varepsilon m^2}\right)^{2k-n} \delta_{(2k-n)\sqrt{\varepsilon}/\lambda}(du).$$

Proof. We choose $c = 0$ and apply

$$\begin{aligned} q(m) &= \int_c^m \frac{d\theta}{v(\theta)} = \frac{\ln(\sqrt{\varepsilon}m + \sqrt{1+\varepsilon m^2})}{\sqrt{\varepsilon}}, \\ g(z) &= q^{-1}(z) = \frac{\sinh(z\sqrt{\varepsilon})}{\sqrt{\varepsilon}}, \\ \int_c^m \frac{\theta}{v(\theta)} d\theta &= \frac{\sqrt{1+\varepsilon m^2} - 1}{\varepsilon}. \end{aligned}$$

Therefore (1.7) gives

$$\int \exp(\lambda u z) C_\lambda(du) = \exp\left(\lambda(\cosh(\sqrt{\varepsilon}z) - 1)/\varepsilon\right) = \sum_{n=0}^{\infty} e^{-\lambda/\varepsilon} \frac{\lambda^n \cosh^n(\sqrt{\varepsilon}z)}{\varepsilon^n n!}.$$

Thus

$$C_\lambda(du) = e^{-\lambda/\varepsilon} \sum_{n=0}^{\infty} \frac{\lambda^n}{(2\varepsilon)^n n!} \sum_{k=0}^{n} \binom{n}{k} \delta_{(2k-n)\sqrt{\varepsilon}/\lambda}(du)$$

is just the compound $\frac{1}{2}\left(\delta_{-\sqrt{\varepsilon}/\lambda} + \delta_{\sqrt{\varepsilon}/\lambda}\right)$-Poisson law. Using the transform equation (1.6) we establish (2.11). $\square$

2.2 A Rational Variance Function

Letac and Mora [29, page 15] indicate that for $p_j > 0$ the variance function

$$V(m) = \frac{m}{(1-m/p_1)(1-m/p_2)\dots(1-m/p_k)}$$

corresponds to a discrete infinitely divisible exponential family which is difficult to determine explicitly. Here we consider $v(m) = m/(1-m)$ which by dilation answers the question for $k = 1$.

In this case ξ and η of (1.8) with $c = 1/2$ are

$$\xi(m) = 2\sqrt{e}me^{-m}, \quad \exp\left(\eta(\xi(m))\right) = \exp\left(m - \frac{1}{2}\,m^2 - \frac{3}{8}\right) = \exp\left(-\frac{1}{2}\,(m-1)^2 + \frac{1}{8}\right).$$

With $\phi(z) = e^z/(2\sqrt{e})$, $f(m) = \exp\left(-\frac{\lambda}{2}(m-1)^2 + \lambda/8\right)$ in (1.16) we conclude that

$$(2.12) \quad e^{\lambda\eta(\xi)} = e^{-3\lambda/8} + \sum_{n=1}^{\infty} \frac{e^{\lambda/8}\xi^n}{2^n e^{n/2} n!} \left[\frac{d^{n-1}}{dx^{n-1}} e^{nx} \frac{d}{dx} \exp(-\lambda(x-1)^2/2)\right]_{x=0}$$

$$= e^{-3\lambda/8} + e^{\lambda/8} \sum_{n=1}^{\infty} \frac{\xi^n}{2^n e^{n/2} n!} \sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k} \left[\frac{d^{k+1}}{dx^{k+1}} \exp\left(-\lambda(x-1)^2/2\right)\right]_{x=0}.$$

For $a > 0$ we have

$$\left[\frac{d^k}{dx^k}\, e^{-a(x-1)^2}\right]_{x=0} = e^{-a} a^{k/2} H_k(\sqrt{a}),$$

where

$$H_n(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j n!}{j!(n-2j)!}\, x^{n-2j}$$

are Hermite polynomials. Therefore (1.10) gives the following.

Theorem 2.5. *For $\lambda > 0$, the natural exponential family with the variance function*

$$V(m) = \frac{m}{\lambda(1-m)}$$

defined on $0 < m < 1$, is generated by the infinitely divisible discrete probability law $\mu_\lambda(du) = \sum_{n=0}^{\infty} \phi_n(\lambda)\delta_n(du)$ with

$$\phi_0(\lambda) := \exp(-3\lambda/8),$$

$$(2.13) \quad \phi_n(\lambda) = \frac{e^{-3\lambda/8}}{2^n e^{n/2} n!} \sum_{k=0}^{n-1} \binom{n-1}{k} n^{n-1-k} \left(\frac{\lambda}{2}\right)^{(k+1)/2} H_{k+1}(\sqrt{\lambda/2}).$$

We note that by [29, Corollary 3.3] applied to the interval $M_F = (0, 1)$ we have $\phi_n(\lambda) > 0$ for all $\lambda > 0$.

Remark 2.6. One can also write (2.13) as

$$\phi_n(\lambda) = \frac{e^{-3\lambda/8}}{2^{n+1}e^{n/2}n!}\sum_{k=0}^{n}\binom{n}{k} n^{n-k}\left(\frac{\lambda}{2}\right)^{(k-1)/2} H_{k+1}(\sqrt{\lambda/2}).$$

Similar calculations for $v(m) = m/(1+m)$ lead to $\phi_4(1) = -\sqrt{e}/64$, so this is not a variance function.

2.3 Positivity of $W_\lambda(m, du)$

It is important to note that a given $v(m)$ does not necessarily determine a distribution regardless of the choice of $\lambda > 0$. Ismail gave such example in [20]. In this section we elaborate on this example and on another example of the form (2.1).

Example 2.1. Let $v(m) = m\sqrt{1-m}, m \in (0,1)$. With $c = 1/2$ we find that

$$\xi(m) = m\left[\frac{1-\sqrt{1-m}}{(\sqrt{2}-1)m}\right]^2, \quad \eta(\xi) = \sqrt{2} - 2\sqrt{1-m}.$$

With $C := (\sqrt{2}-1)^2$ we have that

$$\eta(\xi) = \sqrt{2} - 2 + \frac{4C\,\xi}{1+C\,\xi}.$$

Therefore (1.10) becomes

$$\exp\left(\lambda(\sqrt{2}-2) + 4\lambda C\xi/(1+C\xi)\right) = \sum_{n=0}^{\infty}\phi_n(\lambda)\xi^n. \tag{2.14}$$

The information recorded so far is from [20]. Comparing (4.1) and (10.2.17), page 189 in [14] we see that

$$\begin{aligned}\phi_0(\lambda) &= \exp\left(\lambda(\sqrt{2}-2)\right), \\ \phi_n(\lambda) &= -4\lambda(-C)^n \exp\left(\lambda(\sqrt{2}-2)\right) L_{n-1}^{(-1)}(4\lambda), \quad n > 0,\end{aligned} \tag{2.15}$$

where $L_n^{(-1)}(x)$ is the Laguerre polynomial. Now (1.9) shows that W_λ is a probability distribution if and only if $\phi_n(\lambda) \geq 0$ at the special value of λ under consideration. On the other hand Fejér's formula [35, Theorem 8.22.1] shows that $L_n^{(-1)}(4\lambda)$ is oscillatory at large n for any fixed positive λ. Thus there is no λ for which W_λ is a probability distribution. This is an instance of the usefulness of having the parameter λ.

Example 2.2. Let us now consider the case

$$v(m) = \sqrt{1-m^2}.$$

We take $c = 0$. Thus $q(m) = \arcsin m$, $g(z) = \sin z$, and $\int_0^m \frac{tdt}{v(t)} = 1 - \sqrt{1-m^2}$. To determine $C_\lambda(u)$ we need to invert

$$\exp(\lambda(1-\cos z)) = \int_{\mathbb{R}} C_\lambda(u)\, e^{\lambda u z}\, du,$$

for all z, Re $z \in$ Range of $q(t)$, $t \in (-\pi/2, \pi/2)$. Formula (46) page 55 of [14] is

$$\int_0^\infty K_{ix}(a)\cos(xy)dx = \frac{\pi}{2}\, e^{-a\cosh y}. \tag{2.16}$$

For large p and fixed a, (19) page 88 in [14] is

$$K_{ip}(a) = \frac{\sqrt{2}\,\exp(-p\pi/2)}{(p^2-a^2)^{1/4}}(1+o(1)).$$

Therefore (2.16) gives

$$\frac{1}{\pi}\int_{\mathbb{R}} K_{ix}(\lambda)e^{ixy}dx = e^{-\lambda\cos y}. \tag{2.17}$$

This implies

$$W_\lambda(m,u) = \frac{\lambda}{\pi}K_{i\lambda u}(\lambda)\exp\left(\lambda\sqrt{1-m^2} - \lambda u \arcsin m\right). \tag{2.18}$$

Note that $K_{i\lambda u}(\lambda)$ is real since $K_\nu(x) = K_{-\nu}(x)$ but it fails to be positive for any $\lambda > 0$. Indeed, the second derivative $\frac{d^2}{dy^2}$ of the right hand side of (2.17) at $y = 0$ fails to be negative as it equals $\lambda e^{-\lambda}$.

3 q-Exponential Families with $|q| < 1$

Recall that for $-1 < q < 1$ the q-differentiation operator is

$$(D_{q,x}f)(x) := \frac{f(x) - f(qx)}{x - qx} \quad \text{for } x \neq 0.$$

The q-analogue of the differential equation (1.2) is

$$D_{q,m}w(m,u) = w(m,u)\,\frac{u-m}{V(m)}. \tag{3.1}$$

This equivalent to

$$w(m,u) = \frac{w(mq,u)}{1 + m(1-q)(m-u)/V(m)}. \tag{3.2}$$

When $V(0) \neq 0$ we can rescale m and u by a dilation to make $V(0) = 1$. Now (3.2) has the solution

$$w(m,u) = C(u)\prod_{n=0}^{\infty}\frac{V(q^n m)}{V(m) + m(1-q)(m-u)}, \tag{3.3}$$

provided that the infinite products converge.

For compactly supported measures, the following extends the notion of exponential family from $q = 1$ to $q \in (-1, 1)$.

Definition 3.1. A family of probability measures

$$\mathcal{F}(V) = \{w(m, u)\mu(du) : \ m \in (A, B)\}$$

is a q-exponential family with the variance function V if

(i) μ is compactly supported,

(ii) $0 \in (A, B)$ and $\lim_{t\to 0} w(t, u) = w(0, u) \equiv 1$ for all $u \in \mathrm{supp}(\mu)$,

(iii) $V > 0$ on (A, B) and (3.1) holds for all $m \neq 0$.

Applying $D_{q,m}$ to both sides of $\int w(m, u)\mu(du) = 1$, from (3.1) we deduce that

$$\int uw(m, u)\mu(du) = m. \tag{3.4}$$

This shows that family $\mathcal{F}(V)$ is parameterized by the mean. Applying $D_{q,m}$ to both sides of (3.4), we deduce that

$$\int (u - m)^2 w(m, u)\mu(du) = V(m). \tag{3.5}$$

Thus V is the variance function for $\mathcal{F}(V)$; compare (1.4).

We now show that quadratic variance functions determine q-exponential families uniquely.

Theorem 3.2. *If $\mathcal{F}(V)$ is a q-exponential family with the variance function*

$$V(m) = 1 + am + bm^2$$

and $b > -1 + \max\{q, 0\}$ then

$$w(m, u) = \prod_{k=0}^{\infty} \frac{1 + amq^k + bm^2q^{2k}}{1 + (a - (1 - q)u)mq^k + (b + 1 - q)m^2q^{2k}} \tag{3.6}$$

and $\mu(du)$ is a uniquely determined probability measure with the absolutely continuous part supported on the interval $\frac{a}{1-q} - \frac{2\sqrt{b+1-q}}{1-q} < u < \frac{a}{1-q} + \frac{2\sqrt{b+1-q}}{1-q}$ and no discrete part if $a^2 < 4b$

We remark that for $b \geq 0$ the above family of laws μ appears in [10] in connection to a quadratic regression problem. When $q \geq 0$, one could also allow $b = -1/[N]_q$ for some integer $N \geq 1$ yielding a discrete measure μ supported on $N + 1$ points, compare (4.5) when $b = -1$.

Proof of Theorem 3.2. We rewrite (3.1) as

$$w(m,u) = \frac{V(m)}{V(m)-(1-q)(u-m)m} w(qm,u). \tag{3.7}$$

Thus

$$w(m,u) = w(q^{n+1}m,u)\prod_{k=0}^{n} \frac{1+amq^k+bm^2q^{2k}}{1+(a-(1-q)u)mq^k+(b+1-q)m^2q^{2k}}$$

from which (3.6) follows by taking the limit as $n\to\infty$.

We now recall that for $|t|$ small enough,

$$w(t,u) = \sum_{n=0}^{\infty} \frac{t^n}{[n]_q!} p_n(u), \tag{3.8}$$

is the generating function of the monic Al-Salam–Chihara polynomials

$$up_n(u) = p_{n+1}(u) + a[n]_q p_n(u) + (1+b[n-1]_q)[n]_q p_{n-1}(u). \tag{3.9}$$

This holds because the right hand side of (3.8) satisfies (3.1), see [2]. Since μ is compactly supported, we can integrate (3.8) term by term for $|t|$ small enough; we deduce that $\int p_n(u)\mu(du) = 0$ for all $n\geq 1$. This determines probability measure μ as the measure of orthogonality of polynomials p_n.

Explicit formulas can be read out from [6, Chapter 3], see also [7]. To use these results, we reparameterize (3.9) as follows. Let $\widetilde{p}_n(x) = \alpha^{-n}p_n(\alpha x+\beta)$ with $\alpha = \frac{\sqrt{b+1-q}}{\sqrt{1-q}}$, $\beta = a/(1-q)$. Then $\widetilde{p}_n(x)$ satisfy the three step recurrence

$$(x-\widetilde{a}q^n)\widetilde{p}_n(x) = \widetilde{p}_{n+1}(x) + (1-\widetilde{b}q^{n-1})[n]_q\widetilde{p}_{n-1}(x)$$

with $\widetilde{a} = -\frac{a}{\sqrt{1-q}\sqrt{b+1-q}}$, $\widetilde{b} = \frac{b}{b+1-q}$.

□

The technique we used in the proof of will not work beyond polynomials of degree at most 2. Al-Salam and Chihara [3] proved that the only orthogonal polynomials $\{p_n(x)\}$ with the generating function

$$\sum_{n=0}^{\infty} p_n(x)t^n = A(t)\prod_{n=0}^{\infty}\frac{1-axH(tq^k)}{1-bxK(tq^k)} \tag{3.10}$$

where A,H,K are formal power series with

$$A(t) = \sum_{n=0}^{\infty} a_nt^n, \quad H(t) = \sum_{n=1}^{\infty} h_nt^n, \quad K(t) = \sum_{n=1}^{\infty} k_nt^n \tag{3.11}$$

with $a_0h_1k_1\neq 0$ and $|a|+|b|\neq 0$ are the Al-Salam–Chihara polynomials if $ab=0$ and the q-Pollaczek polynomials if $ab\neq 0$. Theorem 3.2 corresponds to $a=0$. The q-Pollaczek polynomials are in [11].

(A related result appears also in [5, Theorem 23].)

4 Free Exponential Families

The case $q = 0$ can be analyzed more directly. Since it is related to free convolution of measures, it is of interest to elaborate explicitly on the details.

4.1 Free Convolution of measures

We recall the analytic definition of the free convolution of compactly supported probability measures due to Voiculescu [36], see also [37, Section 2.4], [19, Chapter 3]. The Cauchy-Stieltjes transform

$$G_\mu(z) := \int \frac{1}{z-u}\mu(du) \tag{4.1}$$

of a probability measure μ is analytic in $\Re z > 0$. It is known that its inverse $G^{-1}(z)$ exists for $|z|$ large enough. The R-transform of μ defined as $R_\mu(z) = G^{-1}(z) - 1/z$ plays the role of the cumulant generating function. A probability measure μ is the free addictive convolution of probability measures μ_1, μ_2 if

$$R_\mu(z) = R_{\mu_1}(z) + R_{\mu_2}(z).$$

We write $\mu = \mu_1 \boxplus \mu_2$.

The free cumulants of μ are the coefficients of the expansions

$$R_\mu(z) = \sum_{n=1}^{\infty} k_n(\mu) z^{n-1}. \tag{4.2}$$

4.2 Exponential Families with $q = 0$

As previously, we consider $A < 0 < B$ and assume that $V > 0$ on (A, B).

Definition 4.1. A free exponential family with the variance function $V(m) > 0$ in a neighborhood of 0 is a family of probability measures of the form

$$\mathcal{F}(V) := \left\{ \frac{V(m)}{V(m) + m(m-u)}\mu(du) : m \in (A, B) \right\}, \tag{4.3}$$

where μ is a compactly supported probability measure.

It is easy to verify that (4.3) defines a family of measures which fulfills all the requirements of Definition 3.1, including equation (3.1) with $q = 0$. It is also clear that the interval (A, B) must be chosen so that the integral (4.3) converges.

For the purpose of determining measure μ alone, the role of the interval (A, B) is insignificant. Namely, if V is a real analytic function at 0, then μ is determined uniquely by V. Indeed, since $V(0) \neq 0$, the Cauchy-Stieltjes transform (4.1) is well defined for all real $z = m + \frac{V(m)}{m}$ large enough, i.e. for all m close enough to 0, and is given by

$$G_\mu(z) = \frac{m}{V(m)}. \tag{4.4}$$

This determines $G_\mu(z)$ uniquely as an analytic function outside of the support of μ.

In particular, with $V(m) = 1 + am + bm^2$, equation $z = m + \frac{V(m)}{m}$ can be solved for m, giving

$$m = \frac{z - a - \sqrt{(a - z)^2 - 4\,(1 + b)}}{2\,(1 + b)},$$

and

$$G(z) = \frac{a + z + 2\,b\,z - \sqrt{(a - z)^2 - 4\,(1 + b)}}{2\,(1 + az\ + b\,z^2)}.$$

This Cauchy-Stieltjes transform appears in [9, (2)] in a non-commutative quadratic regression problem. It also appears in [4, Theorem 4], [34], and [10, Theorem 4.3]. The corresponding laws are the free-Meixner laws

$$\mu(du) = \frac{\sqrt{4(1+b) - (u-a)^2}}{2\pi(bu^2 + au + 1)} 1_{(a-2\sqrt{1+b}, a+2\sqrt{1+b})} du + p_1\delta_{u_1} + p_2\delta_{u_2}. \tag{4.5}$$

The discrete part of μ is absent except for the following cases:

(i) if $b = 0, a^2 > 1$, then $p_1 = 1 - 1/a^2$, $u_1 = -1/a$, $p_2 = 0$.

(ii) if $b > 0$ and $a^2 > 4b$, then $p_1 = \max\left\{0, 1 - \frac{|a| - \sqrt{a^2-4b}}{2b\sqrt{a^2-4b}}\right\}$, $p_2 = 0$, and $u_1 = \pm\frac{|a|-\sqrt{a^2-4b}}{2b}$ with the sign opposite to the sign of a.

(iii) if $-1 \le b < 0$ then there are two atoms at

$$u_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2b},\ p_{1,2} = 1 + \frac{\sqrt{a^2 - 4b} \mp a}{2b\sqrt{a^2 - 4b}}.$$

This proves the free version of [23, Theorem 3.3], see also [31, Section 4].

Theorem 4.2. *The free exponential family with the variance function*

$$V(m) = 1 + am + bm^2$$

and $b > -1$ *consists of probability measures* (4.3) *with* μ *given by* (4.5)*.*

We remark that if $b \ge 0$ then μ is infinitely divisible with respect to free convolution. In particular, up to dilation and convolution with degenerate δ_m (i.e. up to "type") measure μ is

(i) the Wigner's semicircle (free Gaussian) law if $a = b = 0$; see [37, Section 2.5];

(ii) the Marchenko-Pastur (free Poisson) type law if $b = 0$ and $a \neq 0$; see [37, Section 2.7];

(iii) the free Pascal (negative binomial) type law if $b > 0$ and $a^2 > 4b$; see [34, Example 3.6];

(iv) the free Gamma type law if $b > 0$ and $a^2 = 4b$; see [9, Proposition 3.6];

(v) the free analog of hyperbolic type law if $b > 0$ and $a^2 < 4b$; see [4, Theorem 4];

(vi) the free binomial type law if $-1 \leq b < 0$; see [34, Example 3.4].

We conclude this section with the free version of Proposition 1.1. Recall that the λ-fold free convolution $\mu^{\boxplus\lambda}$ is well defined for the continuous range of values $\lambda \geq 1$, see [32]. Let μ_λ be the dilation by $\lambda \geq 1$ of the free convolution power $\mu^{\boxplus\lambda}$.

Proposition 4.3. *If a compactly supported probability measure μ generates the free exponential family* (4.3) *with the real-analytic variance function $V > 0$ on (A, B), then for all $\lambda \geq 1$ measures μ_λ generate the free exponential family with the variance function $V(m)/\lambda$.*

Moreover, if for every $0 < \lambda < 1$ there is a μ_λ which generates the free exponential family with the variance function $V(m)/\lambda$, then μ is infinitely divisible with respect to the free convolution.

Proof. From (4.4) we determine

$$R_\mu\left(\frac{m}{V(m)}\right) = m, \tag{4.6}$$

which determines the R-transform R_μ uniquely in a neighborhood of 0. Repeating the same calculation with the R-transform R_{μ_λ} of measure μ_λ, we get

$$R_{\mu_\lambda}\left(\frac{m}{V(m)}\right) = \lambda m.$$

Thus

$$\int \frac{V(m)}{V(m) + \lambda m(m-u)} \mu_\lambda(du) = 1$$

for all $|m|$ small enough, and μ_λ generates the corresponding free family (4.3).

The second part follows from the relation

$$R_{\mu_\lambda}\left(\frac{m}{V(m)}\right) = \lambda R_\mu\left(\frac{m}{V(m)}\right),$$

which proves that μ is infinitely divisible with respect to the free convolution. $\square$

Remark 4.4. Combining (4.6) with (1.16) we see that the free exponential family with the analytic variance function V is defined by the unique centered probability measure μ with free cumulants

$$k_{n+1}(\mu) = \left[\frac{1}{n!}\frac{d^{n-1}}{dt^{n-1}} V^n(t)\right]_{t=0},\ n = 1, 2, \ldots.$$

5 q-Exponential Families for $q > 1$

In this section it is convenient to set $q = \frac{1}{p}$ with $0 < p < 1$, and to use again auxiliary parameter $\lambda > 0$. The q-analogue of the differential equation (1.2) is

$$D_{q,m} W_\lambda(m, du) = W_\lambda(m, du) \frac{\lambda(u-m)}{v(m)}. \tag{5.1}$$

As previously, from $\int_{\mathbb{R}} W_\lambda(m, du) = 1$ we deduce by q-differentiation that $\int_{\mathbb{R}} u W_\lambda(m, du) = m$ and $\int_{\mathbb{R}} (u-m)^2 W_\lambda(m, du) = v(m)/\lambda$.

With

$$\lambda_1 := \lambda(1-p)$$

we find that (5.1) is

$$W_\lambda(m, du) = W_\lambda(pm, du) \left[1 + \frac{\lambda_1(u-pm)m}{v(pm)}\right]. \tag{5.2}$$

We first consider (5.2) when $q = \infty$. The general case of $v(0) \neq 0$ reduces to the case $v(0) = 1$. Substituting $p = 0$ into (5.2) we get

$$W_\lambda(m, du) = (1 + \lambda m u)\, C_\lambda(du),$$

which is an analog of equation (1.2) corresponding to $q = \infty$. From this, it is clear that any probability measure $C_\lambda(du)$ such that $\int u C_\lambda(du) = 0$, $\int u^2 C_\lambda(du) = 1/\lambda$ determines its own q-exponential family as long as $1 + \lambda u m \geq 0$ on the support of $C_\lambda(du)$. Moreover, it is easy to see that the only choice of $v(m)$ is a quadratic polynomial $v(m) = 1 + bm - \lambda m^2$ where $b = la^2 \int u^3 C_\lambda(du)$. This show that q-exponential families for $q = \infty$ are not determined uniquely by their variance functions.

It is plausible that non-uniqueness persists for all $q > 1$. For example, the q-Hermite polynomials $\{h_n(x|q)\}$ in [22] correspond to probability measures which are not determined uniquely by moments. The N-extremal solutions of the moment problem, [1], are given by a one-parameter family $\{\mu(du; a) : a \in (p, 1)\}$ which is completely characterized by Ismail and Masson in [22], see also Chapter 21 in [21]. Unfortunately, the construction of the corresponding exponential family via equation (5.1) led us to the family of measures

$$W_\lambda(m, du) = \prod_{k=0}^{\infty} (1 + \lambda_1 m u/q^k - \lambda_1 m^2/q^{2k}) \mu(du; a) \tag{5.3}$$

with negative densities.

The non-uniqueness within the class of quadratic variance functions $v(m)$ is confirmed by the following two examples.

Example 5.1. Consider the absolutely continuous family with support in $(0, \infty)$ with the density

$$w_\lambda(m, u) = \frac{(p^{-\lambda} - 1)^\lambda \, (p; p)_\infty \sin(\pi\lambda)}{\pi m^\lambda \, (p^{1-\lambda}; p)_\infty} \frac{u^{\lambda-1}}{(-u(p^{-\lambda} - 1)/m; p)_\infty}. \tag{5.4}$$

This is the case of p-Laguerre polynomials [25, §3.21]. With $q = 1/p$, a calculation verifies that

$$D_{q,m} w_\lambda(m,u) = \frac{p(1-p^\lambda)}{m^2(1-p)}\, w_\lambda(m,u)(u-m). \tag{5.5}$$

Now (3.21.2), page 108 of [25] shows that

$$\int_0^\infty w_\lambda(m,u)\,du = 1, \quad \int_0^\infty w_\lambda(m,u)\,u\,du = m. \tag{5.6}$$

(The latter integral follows also from the former by q-differentiation and (5.5).) Thus

$$\mathcal{F} = \{w_\lambda(m,u)1_{u>0}du : m > 0\}$$

is parameterized by the mean. Applying $D_{q,m}$ again, we get the variance function

$$V(m) = \int_0^\infty (u-m)^2\, W_\lambda(m,du) = \frac{m^2}{\lambda_q} \tag{5.7}$$

with $\lambda_q = \frac{1-p}{p(1-p^\lambda)}$. This is a continuous q-analogue of the gamma family with $v(m) = m^2$.

Example 5.2. For $m > 0$, consider the family of discrete measures

$$W_\lambda(m,du) = w_\lambda(m,u)\mu(du)$$

with the density

$$w_\lambda(m,u) = u^\lambda \frac{(-c, -p/c; p)}{(-cu, -cp^\lambda, -c^{-1}p^{-la+1}; p)}, \; c = (p^{-\lambda} - 1)/m \tag{5.8}$$

with respect to discrete measure

$$\mu(du) = \frac{(p^\lambda; q)_\infty}{(p;p)_\infty} \sum_{n=-\infty}^{\infty} \delta_{p^n}(du).$$

This is again related to p-Laguerre polynomials [25, §3.21]. With $q = 1/p$, a calculation verifies that (5.5) holds.

Now (3.21.3), page 108 of [25] shows that

$$\int_{\mathbb{R}} w_\lambda(m,u)\,\mu(du) = 1. \tag{5.9}$$

As previously, applying $D_{q,m}$ to both sides of (5.9) and using (5.5) we get

$$\int_{\mathbb{R}} u w_\lambda(m,u)\,\mu(du) = m. \tag{5.10}$$

Applying $D_{q,m}$ to both sides of (5.10) and using (5.5) again, we get $V(m) = m^2/\lambda_q$, compare (5.7). Thus $\{W_\lambda(m,du) : \; m > 0\}$ is a discrete q-analogue of the gamma exponential family; it shares the variance function and the q-differential equation with the continuous q-analogue of the gamma exponential family from the previous example.

6 Shifted q-Exponential Families

The special role played by 0 in Definition 3.1 is due to the fact that q-derivative $D_{q,x}$ is dilation invariant but not translation invariant. More generally, we consider the L-operator introduced by Hahn [18]. This is a q-differentiation operator centered at $\theta \in \mathbb{R}$, which we can write as

$$({}_\theta\widetilde{D}_{q,x} f)(x) = \frac{f(x) - f(qx + (1-q)\theta)}{(1-q)(x-\theta)},\ x \neq \theta,\ q \neq 1. \tag{6.1}$$

The usual q-derivative $D_{q,x}$ corresponds to $\theta = 0$. For $\theta \neq 0$ a dilation reduces all such operators to $\theta = 1$, in which case we use shorter notation

$$\widetilde{D}_{q,x} := {}_1\widetilde{D}_{q,x}. \tag{6.2}$$

Note that

$$\widetilde{D}_{q,x} 1 = 0,\ \widetilde{D}_{q,x} x = 1.$$

With $A \leq 1 \leq B$ and $V > 0$ on (A, B), the shifted q-exponential family with variance function V is the family of probability measures

$$\mathcal{F}_\theta = \{w(m,u)\,\mu(du) : m \in (A,B)\}$$

such that

$$\widetilde{D}_{q,m} w(m,u) = w(m,u)\frac{u-m}{V(m)}.$$

In this discussion we are less restrictive than in Definition 3.1: in the admissible range of values of θ we include the end-points of (A, B), and we allow non-compact support for μ. Such a generalization is beyond the scope of this paper, so we give only one explicit example for $0 < q < 1$, $B = 1$ and one for $q > 1$, $A = 1$.

Example 6.1. Consider the case of the Wall polynomials, see [25, §3.20]. In this case we have a family of discrete probability measures

$$W(m, du) = w(m,u)\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n}\,\delta_{q^n}(du),$$

where the density is

$$w(m,u) = a^{\ln u/\ln q}(aq;q)_\infty,\ a = (1-m)/q. \tag{6.3}$$

From (3.20.3) in [25] we see that $\int_{\mathbb{R}} p_1(u; a|q) w(m,u) du = 0$, which implies $\int_{\mathbb{R}}(1 - ap - u)w(m,u)du = 0$. Hence the family

$$\mathcal{F} = \{W(m,du) :\ 0 < m < 1\}$$

is again parameterized by the mean,

$$\int_{\mathbb{R}} uW(m,du) = m.$$

From [25, (3.20.3)] we calculate the variance function

$$V(m) = m(1-m)(1-q).$$

Now

$$\widetilde{D}_{q,m} w(m,u) = \frac{u-m}{(1-q)m(1-m)}\, w(m,u). \tag{6.4}$$

Thus (6.3) defines a shifted analog of the q-Binomial family. Note that although equation (6.4) makes sense also for $q=0$, it then gives a degenerated law δ_1, not the translation of a free binomial law.

Example 6.2. For $0<p<1$, let $q=1/p$ and consider the Al-Salam–Carlitz polynomials $\{V_n^{(a)}(x;p)\}$, [25, §3.25]. Let

$$W(m,du) = w(m,u)\mu(du) = a^{-\ln u/\ln p}(aq/u;p)_\infty \mu(du), \tag{6.5}$$

where $\mu(du) = \sum_{n=0}^{\infty} p^{n^2}/(p;p)_n \delta_{p^{-n}}(du)$ and $m=a+1$. Now with $q=1/p$ we have

$$\widetilde{D}_{q,m} w(m,u) = \frac{w(m,u)}{(m-1)(1-1/p)}\{1-u(1-(m-1)/u)\},$$

which simplifies to

$$\widetilde{D}_{q,m} w(m,u) = \frac{p}{1-p}\, w(m,u)\frac{u-m}{m-1}. \tag{6.6}$$

Since [25, formula (3.25.2)] implies $\int_{[0,\infty)} W(m,du) = 1$, therefore applying $\widetilde{D}_{q,m}$ and taking (6.6) into account we deduce $\int_{[0,\infty)} uW(m,du) = m$. Similarly $V(m) = (1-p)(m-1)/p$. Thus (6.5) defines the family of measures

$$\mathcal{F} = \{W(m,du) :\ m>1\},$$

which is a shifted q-analogue of the Poisson exponential family with $q=1/p>1$.

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